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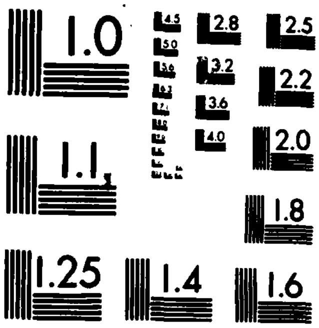

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EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS STEADY NAVIER-STOKES EQUATIONS

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EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS STEADY NAVIER-STOKES EQUATIONS

Charles J. Amick

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ABSTRACT

This paper concerns the existence of steady solutions to the Navier-Stokes equations in a bounded domain $\Omega \subset R^2$. The condition of solenoidality for the velocity field imposes a necessary condition on the boundary data. For a certain class of symmetrical domains, ~~we~~ the authors show that this necessary condition implies the existence of a solution to the problem. The method consists of proving a priori bounds on solutions by assuming the contrary, rescaling the equations, and then arriving at a solution to the steady Euler equations in the limit. Examination of this equation leads to the desired contradiction. After one has suitable bounds on any solutions, one uses the Leray-Schauder theorem to prove existence.

In addition, ~~we~~ The authors remark on the problem of a general bounded domain ~~$\Omega \subset R^2$~~ and suggest how certain maximum principles might yield the expected results.

AMS (MOS) Subject Classifications: 35Q10, 76D05, 35B50

Key Words: Navier-Stokes equations, a priori bounds, maximum principles

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The steady Navier-Stokes equations model the flow of a viscous, incompressible fluid in some region Ω. With u and p denoting the velocity and pressure, respectively, one has

$$\left.\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u &= f - \nabla p \\ \nabla\cdot u &= 0 \end{aligned}\right\} \text{ in } \Omega \; . \qquad (*)$$

$$u = \tilde{g} \text{ on } \partial\Omega \; ,$$

and the problem is to solve for (u,p) for prescribed constant viscosity $\nu > 0$, external force f, and boundary data $\tilde{g}$. The boundary data is not arbitrary but satisfies the necessary condition

$$0 = \int_\Omega \nabla\cdot u = \sum_{i=1}^{m} \int_{\Gamma_i} \tilde{g}_i\cdot n_i \qquad (**)$$

where Γ_i denote the components of $\partial\Omega$ and n_i denotes the outward normal. Condition (**) merely states that the total outflow across $\partial\Omega$ is zero. If one imposes the stronger condition of no outflow across each boundary component:

$$\int_{\Gamma_i} \tilde{g}_i\cdot n_i = 0, \; i = 1,\ldots,m \quad , \qquad (***)$$

then classical theory going back to Leray fifty years ago ensures that (*) has a solution for any $\nu > 0$ and any suitably smooth function f.

If (***) is replaced by the necessary condition (**), the standard methods only give the existence of a solution when ν is sufficiently large. In this paper, we show that the necessary condition on $\tilde{g}$ yields a solution (u,p) to (*) in a certain class of domains Ω in the plane. We also remark on the case of a general domain in the plane, and show how certain maximum principles are applicable.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS STEADY NAVIER-STOKES EQUATIONS

Charles J. Amick

1. INTRODUCTION

In this paper we consider the steady Navier-Stokes equations in a bounded domain (an open, connected set) $\Omega \subset \mathbb{R}^2$:

$$\left.\begin{aligned} -\nu_o \Delta u + (u\cdot\nabla)u &= f - \nabla p \,, \\ \nabla\cdot u &= 0 \,, \end{aligned}\right\} \text{ in } \Omega \qquad \begin{aligned}&(1.1)\\&(1.2)\end{aligned}$$

$$u = \tilde{g} \quad \text{on } \partial\Omega \quad . \tag{1.3}$$

We assume that $\partial\Omega$ is infinitely differentiable, $f \in C^\infty(\bar{\Omega} \to \mathbb{R}^2)$, $\tilde{g} \in C^\infty(\partial\Omega \to \mathbb{R}^2)$, and $\nu_o > 0$. The problem is to find a velocity field $u = (u_1,u_2)$ and a pressure p satisfying (1.1) - (1.3). The boundary data $\tilde{g}$ is not arbitrary, but satisfies the compatability condition

$$0 = \int_\Omega \text{div } u = \sum_{i=1}^{m} \int_{\Gamma_i} \tilde{g}_i\cdot n_i \quad , \tag{1.4}$$

where the Γ_i are the components of $\partial\Omega$ and n_i denotes the outward normal to Γ_i. <u>We shall assume throughout this paper that</u> (1.4) <u>holds</u>. Our intention is to prove the existence of a solution (u,p) to (1.1) - (1.3) for any $\tilde{g}$ satisfying the necessary condtion (1.4). The results will hold for a certain class of domains $\Omega \subset \mathbb{R}^2$, and we shall remark on the general case in section 3.

If one replaces (1.4) by the stronger condition

$$\int_{\Gamma_i} \tilde{g}_i\cdot n_i = 0, \; i = 1,\ldots,m \quad , \tag{1.5}$$

then the existence of a solution (u,p) for any $\nu_o > 0$ is classical (at

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least for domains $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3) and goes back to the fundamental work of Leray [9] (see also [2], [8], [12]). Condition (1.5) has been assumed in the recent work of Foias and Temam [3], [4], Saut and Temam [10], and Temam [11], [12] on properties of solutions to the steady Navier-Stokes equations. As we shall see in section 2, the main advantage of (1.5) is that it allows the boundary data $\tilde{g}$ to be extended to Ω as a curl; more precisely, there is a smooth function ψ defined on $\bar{\Omega}$ such that $\nabla \times \psi = \tilde{g}$ on $\partial\Omega$. One then mollifies ψ near to $\partial\Omega$ and obtains a priori bounds suitable for the Leray-Schauder theorem.

If (1.4) holds but not (1.5), then one cannot extend $\tilde{g}$ as a curl to Ω. However, standard theory [8], [12] ensures the existence of a function $g \in C^\infty(\bar{\Omega} \to \mathbb{R}^2)$ satisfying the Stokes equations:

$$\left.\begin{aligned} -\nu_0 \Delta g &= -\nabla q \\ \nabla \cdot g &= 0 \, , \end{aligned}\right\} \text{ in } \Omega \tag{1.6, 1.7}$$

$$g = \tilde{g} \quad \text{on} \quad \partial\Omega \; . \tag{1.8}$$

If we set $u = g+v$, then (1.1) - (1.3) is equivalent to solving the problem

$$\left.\begin{aligned} -\nu_0 \Delta v + (g\cdot\nabla)v + (v\cdot\nabla)v + (v\cdot\nabla)g &= \tilde{f} - \nabla p \, , \\ \nabla\cdot v &= 0 \, , \end{aligned}\right\} \text{ in } \Omega \tag{1.9, 1.10}$$

$$v = 0 \quad \text{on} \quad \partial\Omega \; , \tag{1.11}$$

where $\tilde{f} = f-(g\cdot\nabla)g$ is fixed and we absorbed the q term into p. Let $H(\Omega)$ denote the completion of $\{v = (v_1,v_2) \in C_0^\infty(\Omega\to\mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } \Omega\}$ in the Dirichlet norm:

$$\langle v,v\rangle_H = |v|_H^2 = \sum_{i,j=1}^{2} \int_\Omega \left|\frac{\partial v_i}{\partial x_j}\right|^2 \; .$$

The problem of solving (1.9) - (1.11) for (v,p) is equivalent to finding a (weak) solution v of the following equation:

$$\nu_o\langle\phi,v\rangle_H + \int_\Omega \phi\cdot(g\cdot\nabla)v + \int_\Omega \phi\cdot(v\cdot\nabla)v + \int_\Omega \phi\cdot(v\cdot\nabla)g$$
$$= \int_\Omega \phi\cdot\tilde{f} \quad \text{for all} \quad \phi \in H(\Omega) \quad . \tag{1.12}$$

In the usual way [8], [12], this is equivalent to solving an operator equation

$$\dot{\nu}_o v = Av + F \quad , \tag{1.13}$$

where $F \in H(\Omega)$ and A is a compact map of $H(\Omega)$ into itself. Here F and A are defined by the Riesz representation theorem:

$$\int_\Omega \phi\cdot(g\cdot\nabla)v + \int_\Omega \phi\cdot(v\cdot\nabla)v + \int_\Omega \phi\cdot(v\cdot\nabla g) = -\langle\phi,Av\rangle_H$$

and

$$\int_\Omega \phi\cdot\tilde{f} = \langle\phi,F\rangle_H$$

for all $\phi \in H(\Omega)$. After one has found a solution $v \in H$ to (1.13), it is then standard to recover a smooth solution (p,v) of (1.9) - (1.11).

For each $\nu > \nu_o$, let

$$S(\nu) = \{v \in H(\Omega) : \nu v = Av + F\} \quad . \tag{1.14}$$

Note that $v \in S(\nu)$ if and only if there is a pressure p such that $(g+v,p)$ satisfy (1.1) - (1.3) with ν_o replaced by ν. We shall prove in Theorem 2.1 that $S(\nu)$ is non-empty if ν exceeds some critical value $\tilde{\nu}$ and that

$$|v|_H < \text{const.}/\nu, \quad \nu > \tilde{\nu} \quad . \tag{1.15}$$

Our main result in section 2 is the following estimate:

$$\sup_{\nu\in[\nu_o,\tilde{\nu}]} \; \sup_{v\in S(\nu)} |v|_H < \infty \quad . \tag{1.16}$$

This ensures that all solutions of the equation

$$v = \frac{\lambda}{\nu_o}(Av + F) \quad , \tag{1.17}$$

$\lambda \in [0,1]$, are uniformly bounded in $H(\Omega)$, and so the existence of a solution to (1.13) follows by the Leray-Schauder theorem. We shall prove (1.16) by assuming it is false, and then deriving a contradiction. Although (1.15) holds for bounded domains in $\mathbb{R}^2$ or $\mathbb{R}^3$, we have only succeeded in proving (1.16) for a certain class of domains $\Omega \subset \mathbb{R}^2$ and data $\tilde{g}$ and f.

Since (1.4) and (1.5) are the same for $m = 1$, we shall restrict attention at all times to domains with $m > 2$ boundary components.

DEFINITION: *A bounded domain* $\Omega \subset \mathbb{R}^2$ *is said to be admissible if* (a) $\partial\Omega$ *is of class* C^∞, (b) $\partial\Omega$ *consists of* $m > 2$ *components* Γ_i, (c) Ω *is symmetric about the line* $\{x_2 = 0\}$ *and* (d) *each component* Γ_i *intersects the line* $\{x_2 = 0\}$.

A function $h = (h_1, h_2)$ mapping Ω or $\partial\Omega$ into $\mathbb{R}^2$ is said to be symmetric about the line $\{x_2 = 0\}$ if h_1 is an even function of x_2 while h_2 is an odd function of x_2.

DEFINITION: *A pair* $(f, \tilde{g})$ *is said to be admissible data if* (a) $f \in C^\infty(\bar{\Omega} \to \mathbb{R}^2)$, (b) $\tilde{g} \in C^\infty(\partial\Omega \to \mathbb{R}^2)$, *and* (c) f *and* $\tilde{g}$ *are symmetric about the line* $\{x_2 = 0\}$.

If Ω is an admissible domain and $(f, \tilde{g})$ is admissible data, it is natural to seek a solution (u, p) to (1.1) - (1.3) with u symmetric about $\{x_2 = 0\}$; the corresponding pressure p will be an even function of x_2. The main result of this paper is the following theorem which is proved in section 2.

THEOREM 1.1. *Let* $\Omega \subset \mathbb{R}^2$ *be an admissible domain and let* $(f, \tilde{g})$ *be admissible data. Then for every* $\nu_o > 0$ *there exists a solution* $(u, p) \in C^\infty(\bar{\Omega} \to \mathbb{R}^2) \times C^\infty(\bar{\Omega} \to \mathbb{R})$ *of* (1.1) - (1.3). *The function* u *is symmetric about* $\{x_2 = 0\}$ *and the pressure* p *is an even function of* x_2.

Acknowledgement. I am indebted to Professor J. Heywood for bringing this problem to my attention. In addition, he strongly suggested that the proof of (1.16) might follow by assuming the contrary and deriving a contradiction.

2. A PRIORI BOUNDS

Before proceeding to the proof of Theorem 1.1, we show how the condition (1.5) leads to solutions of (1.1) - (1.3) for any $\nu_o > 0$. Assume for the moment that Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with smooth boundary and f and $\tilde{g}$ are smooth on $\bar{\Omega}$ and $\partial\Omega$, respectively. Equation (1.5) allows one to extend the boundary data $\tilde{g}$ to Ω as a curl; $g = \nabla x \psi$ in $\bar{\Omega}$ and $g = \tilde{g}$ on $\partial\Omega$ [2], [8], [12]. For each $\varepsilon > 0$, let $\rho(\cdot;\varepsilon)$ denote a suitable mollifier [2; p. 209], [8; p. 108], [12; p. 175] with $\rho \equiv 1$ near to $\partial\Omega$ and with support in an ε-neighborhood of $\partial\Omega$. Upon choosing ε sufficiently small and setting $g_\varepsilon = \nabla x(\mu(\cdot;\varepsilon)\psi(\cdot))$, one has [2; p. 210], [8; p. 109], [12; p. 175]

$$|\int_\Omega g_\varepsilon \cdot (v\cdot\nabla)v| < \text{const. } \varepsilon |v|_H^2 \quad \text{for all} \quad v \in H(\Omega) \tag{2.1}$$

and the constant is independent of ε and v. One seeks a solution of (1.1) - (1.3) of the form $u = g_\varepsilon + v$, so that v is to satisfy (1.9) - (1.11) (with g replaced by g_ε and $\hat{f}$ replaced by $\hat{f} + \nu_o \Delta g_\varepsilon$), or, equivalently, (1.13). With $S(\nu)$ as in (1.14), it suffices to choose ε such that $\sup\limits_{\nu>\nu_o} \sup\limits_{v\in S(\nu)} |v|_H < \infty$. If $v \in S(\nu)$, then setting $\phi = v$ in (1.12) yields

$$\begin{aligned} \nu\, |v|_H^2 &= -\int_\Omega v\cdot(v\cdot\nabla)g_\varepsilon + \int_\Omega v\cdot\hat{f} \\ &= \int_\Omega g_\varepsilon\cdot(v\cdot\nabla)v + \int_\Omega v\cdot\hat{f} \\ &< \text{const. } \varepsilon|v|_H^2 + \frac{\text{const.}}{\varepsilon}\, |\hat{f}|^2_{L^2(\Omega)} \quad . \end{aligned} \tag{2.2}$$

If we choose ε_o such that const. $\varepsilon_o < \nu_o/2$, then

$$|v|_H^2 < \frac{\text{const.}}{\varepsilon_o \nu_o}\, |\hat{f}|^2_{L^2(\Omega)} \tag{2.3}$$

for all $v \in S(\nu)$ if $\nu > \nu_o$. It then follows from the Leray-Schauder theorem that a solution to (1.13) exists.

The key point in the analysis above was the representation $g = \nabla \times \psi$ which allowed us to put a mollifier with small support near $\partial\Omega$ inside the curl. The form of g_ε led to (2.1) which was the key step in proving (2.3) from (2.2). If one drops the condition (1.5) (but still has the necessary condition (1.4)), then it is not clear how to proceed. One might set $M = \{g \in C^\infty(\bar{\Omega} \to \mathbb{R}^n) : \nabla\cdot g = 0 \text{ in } \Omega \text{ and } g = \tilde{g} \text{ on } \partial\Omega\}$ and set

$$\mu = \inf_{g\in M} \max_{v\in H(\Omega)} \frac{\int_\Omega g\cdot(v\cdot\nabla)v}{|v|_H^2} .$$

If $\mu < \nu_o$, then one gets a priori bounds for $v \in S(\nu)$, $\nu > \nu_o$, by (2.2). When $\tilde{g}$ satisfies (1.5), then (2.1) gives $\mu = 0$. Unfortunately, in general one will not have $\mu < \nu_o$, and so a different approach must be taken.

The following theorem gives the existence of a unique solution to (1.1) - (1.3) if ν is sufficiently large.

THEOREM 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with $\partial\Omega$ of class C^∞, and let $f \in C^\infty(\bar{\Omega} \to \mathbb{R}^n)$ and $\tilde{g} \in C^\infty(\partial\Omega \to \mathbb{R}^n)$. Then there exists $\tilde{\nu} > 0$ such that (1.1) - (1.3) has a solution $(u,p) \in C^\infty(\bar{\Omega} \to \mathbb{R}^n) \times C^\infty(\bar{\Omega} \to \mathbb{R})$ for all $\nu > \tilde{\nu}$. In addition, (u,p) is unique up to an additive constant for p.

Proof. Let g be as in (1.6) - (1.8). Now

$$\left|\int_\Omega g\cdot(v\cdot\nabla)v\right| \le \text{const.}\ |g|_{L^\infty(\Omega)} |v|_H^2 = C|v|_H^2 \quad , \quad v \in H(\Omega) \quad ,$$

and the constant C is independent of v. Let $\tilde{\nu}$ be any fixed number greater than $2C$. We claim that (1.1) - (1.3) has a solution for any $\nu > \tilde{\nu}$. Indeed, if ν_o is so given, then it suffices to solve (1.13). If $v \in S(\nu)$, then (2.2) gives

$$\nu|v|_H^2 = \int_\Omega g\cdot(v\cdot\nabla)v + \int_\Omega \tilde{f}v \le C\ |v|_H^2 + \text{const}\ |\tilde{f}|_{L^2(\Omega)} |v|_H \quad ,$$

whence

$$|v|_H \leq \frac{\text{const.}}{\nu} \cdot |\tilde{f}|_{L^2(\Omega)} \tag{2.4}$$

for all $\nu > \nu_0$. This estimate gives an *a priori* bound for solutions to (1.17), and so existence follows from the Leray-Schauder theorem.

If (u,p) and (w,q) satisfy (1.1) - (1.3), then a calculation yields

$$\begin{aligned}\nu_0 |u-w|_H^2 &= \int_\Omega w\cdot((u-w)\cdot\nabla)(u-w) \\ &\leq \text{const.}\ |w|_{L^4(\Omega)} \ |u-w|_{L^4(\Omega)} \ |u-w|_H \\ &\leq \text{const.}\ |u-w|_H^2\end{aligned}$$

by (2.4). The constant is independent of ν_0, and so if $\tilde{\nu}$ is sufficiently large, then $\nu_0 > \tilde{\nu}$ ensures a unique solution of (1.13). The existence of a pressure p and the regularity of (u,p) is standard [8], [12], q.e.d.

Throughout the rest of this section, we shall assume that $\Omega \subset R^2$ is an admissible domain and $(f,\tilde{g})$ are admissible data. Let g be a solenoidal extension of $\tilde{g}$ to Ω as in (1.6) - (1.8); since $\tilde{g}$ and Ω are symmetric about $\{x_2 = 0\}$, the extension g may be taken to be symmetric. We seek a solution u of (1.1) - (1.3) of the form $u = g+v$, where $v \in H(\Omega)$ and v is to be symmetric about $\{x_2 = 0\}$. Let $H_s(\Omega)$ denote the closed subspace of $H(\Omega)$ consisting of velocity fields which are symmetric about $\{x_2 = 0\}$. It is a Hilbert space with the previous inner product $\langle\cdot,\cdot\rangle_H$. The problem of solving (1.1) - (1.3) is equivalent to finding $v \in H_s(\Omega)$ satisfying $\nu_0 v = Av + F$, where $F \in H_s(\Omega)$ and A is a compact map of H_s into itself. If we set

$$S(\nu) = \{v \in H_s(\Omega) : \nu v = Av + F\} \quad ,$$

then the proof of Theorem 2.1 ensures that $S(\nu)$ is non-empty for all $\nu > \tilde{\nu}$ and that (1.15) holds. In order to prove (1.16) (and thereby Theorem 1.1), we shall assume the contrary and derive a contradiction.

Let $v_n \in S(\nu_n)$ with $|v_n|_H \to \infty$ as $n \to \infty$. Without loss of generality we may assume that $\nu_n \to \nu \in [\nu_o, \tilde{\nu}]$ as $n \to \infty$. Since $v_n \in S(\nu_n)$, it satisfies (1.12) with ν_o replaced by ν_n. Standard theory gives the existence of a pressure p_n (normalized by $\int_\Omega p_n = 0$) such that

$$\begin{aligned} -\nu_n \Delta v_n + (g\cdot\nabla)v_n + (v_n\cdot\nabla)g + (v_n\cdot\nabla)v_n &= \tilde{f} - \nabla p_n \\ &= f - (g\cdot\nabla)g - \nabla p_n \quad \text{in } \Omega \quad . \end{aligned} \tag{2.5}$$

If we set $u_n = g + v_n$, then

$$-\nu_n \Delta u_n + (u_n\cdot\nabla)u_n = f - \nabla p_n \quad \text{in } \Omega$$

and we have absorbed the term $-\nu_n \Delta g \equiv \frac{-\nu_n}{\nu_o} \nabla q$ into the p_n term. Define

$$\tilde{u}_n = u_n/|v_n|_H, \; \tilde{v}_n = v_n/|v_n|_H \quad \text{and} \quad \tilde{p}_n = p_n/|v_n|_H^2 \quad .$$

Without loss of generality, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H_s(\Omega)$ and $\tilde{v}_n \to \tilde{v}$ in $L^q(\Omega)$, $q \in [1,\infty)$. The estimates of Cattabriga [1], [12] applied to (2.5) yield

$$\begin{aligned} |p_n|_{W^{1,q}(\Omega)} &\le \text{const.}\{\,|\tilde{f}|_{L^q(\Omega)} + |(g\cdot\nabla)v_n + (v_n\cdot\nabla)v_n + (v_n\cdot\nabla)g|_{L^q(\Omega)}\} \\ &\le \text{const.}\{1 + |v_n|_H^2\} \end{aligned}$$

for any $q \in (1,2)$. In particular, $\tilde{p}_n$ is bounded in $W^{1,q}(\Omega)$, $q \in (1,2)$, independently of n, and so $\tilde{p}_n \rightharpoonup \tilde{p}$ in $W^{1,q}(\Omega)$, $q \in (1,2)$, and $\tilde{p}_n \to \tilde{p}$ in $L^q(\Omega)$, $q \in [1,\infty)$. If we multiply (2.5) by v_n, integrate over Ω, and then divide by $|v_n|_H^2$, there results

$$\nu_n = \int_\Omega g\cdot(\tilde{v}_n\cdot\nabla)\tilde{v}_n + \frac{1}{|v_n|_H}\int_\Omega \tilde{f}\cdot\tilde{v}_n \quad . \tag{2.6}$$

Since $\tilde{v}_n \to \tilde{v}$ in $L^2(\Omega)$, we may take the limit of $n \to \infty$ in (2.6):

$$\nu = \int_\Omega g\cdot(\tilde{v}\cdot\nabla)\tilde{v} \quad . \tag{2.7}$$

If we multiply (2.5) by a fixed $\phi \in C_0^\infty(\Omega + \mathbb{R}^2)$, integrate over Ω, divide by $|v_n|_H^2$ and let $n \to \infty$, there results

$$\int_\Omega \phi\cdot(\tilde{v}\cdot\nabla)\tilde{v} = -\int_\Omega \phi\cdot\nabla\tilde{p}, \; \phi \in C_0^\infty(\Omega + \mathbb{R}^2) \quad . \tag{2.8}$$

Since ϕ was arbitrary, it follows that $(\hat{v},\hat{p})$, with $\hat{v} \in H_s(\Omega)$, is a weak solution of the steady Euler equations;

$$(\hat{v}\cdot\nabla)\hat{v} = -\nabla\hat{p} \text{ almost everywhere in } \Omega \quad . \tag{2.9}$$

Recall that the solenoidality of $\hat{v}$ is immediate from its membership in $H_s(\Omega)$ while $\hat{v} = 0$ on $\partial\Omega$ (in the sense of a trace) for the same reason. Since $\hat{v} \in H_s(\Omega)$ and $\hat{p} \in W^{1,q}(\Omega)$, $q \in (1,2)$, it follows that $(\hat{v}\cdot\nabla)\hat{v}$, $\nabla\hat{p} \in L^q(\Omega)$, $q \in (1,2)$, and so (2.8) actually holds for all $\phi \in L^r(\Omega)$, $r \in (2,\infty)$. In particular, setting $\phi = g$ yields

$$\begin{aligned}\int_\Omega g\cdot(\hat{v}\cdot\nabla)\hat{v} &= -\int_\Omega g\cdot\nabla\hat{p} = -\int_\Omega \operatorname{div}(g\hat{p}) \\ &= \int_{\partial\Omega} \hat{p}g\cdot n = \sum_{i=1}^{m} \int_{\Gamma_i} \hat{p}\tilde{g}_i\cdot n_i \quad .\end{aligned} \tag{2.10}$$

The integration by parts in (2.9) is justified since $\hat{p} \in W^{1,q}(\Omega)$ has a well-defined trace on $\partial\Omega$.

To motivate what is about to follow, we argue formally for the moment. Since $\hat{v} \in H_s(\Omega)$, we have $\hat{v} = 0$ on $\partial\Omega$ in the sense of a trace, and so (2.9) suggests that $\hat{p}$ is a constant C_i on each component Γ_i of $\partial\Omega$. If we could show that these constants are all equal to, say, C, then (2.10) would give

$$\int_\Omega g\cdot(\hat{v}\cdot\nabla)\hat{v} = C \sum_{i=1}^{m} \int_{\Gamma_i} \tilde{g}_i\cdot n_i = 0 \tag{2.11}$$

by (1.4). However, (2.7) shows that (2.10) is impossible, and so we will have the desired contradiction. In the following two theorems, we justify these formal arguments.

THEOREM 2.2. <u>The trace of</u> $\hat{p}$ <u>on</u> Γ_i <u>is a constant</u> C_i, $i = 1,\ldots,m$, <u>almost everywhere on</u> Γ_i.

<u>Proof</u>. Let i be arbitrary and set $\Gamma = \Gamma_i$. Let $z_0 \in \Gamma$ and change to a new orthogonal coordinate system $(\hat{x}_1,\hat{x}_2)$ centered at z_0 with the $\hat{x}_2$-axis pointing along the inner normal to Γ at z_0. We write

$\tilde{\nabla} = (\frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2})$. For small $\varepsilon > 0$, the boundary component Γ is given locally by $\tilde{x}_2 = h(\tilde{x}_1)$, $\tilde{x}_1 \in (-\varepsilon,\varepsilon)$, with $h \in C^\infty$. Let $\delta > 0$ be sufficiently small such that

$$A = A(\varepsilon,\delta) = \{(\tilde{x}_1,\tilde{x}_2) : \tilde{x}_1 \in (-\varepsilon,\varepsilon),\ \tilde{x}_2 \in (h(\tilde{x}_1), h(\tilde{x}_1) + \delta)\} \subset \Omega \quad .$$

Now

$$\int_A \frac{|(\tilde{v}\cdot\tilde{\nabla})\tilde{v}|}{|\tilde{x}_2-h(\tilde{x}_1)|} d\tilde{x}_1 d\tilde{x}_2 < \text{const.}\left(\int_{-\varepsilon}^{\varepsilon}\int_{h(\tilde{x}_1)}^{h(\tilde{x}_1)+\delta} \frac{|\tilde{v}(\tilde{x}_1,\tilde{x}_2)|^2}{|\tilde{x}_2-h(\tilde{x}_1)|^2} d\tilde{x}_2 d\tilde{x}_1\right)^{1/2} |\tilde{v}|_H$$

$$< \text{const.}\left(4\int_{-\varepsilon}^{\varepsilon}\int_{h(\tilde{x}_1)}^{h(\tilde{x}_1)+\delta} |\tilde{\nabla}\tilde{v}|^2\right)^{1/2} |\tilde{v}|_H < \text{const.}|\tilde{v}|_H^2 = \text{const.} \quad , \tag{2.12}$$

where we have used the standard estimate

$$\int_a^{a+\delta}\left(\frac{w(\tilde{x}_2)}{\tilde{x}_2-a}\right)^2 d\tilde{x}_2 < 4\int_a^{a+\delta}(w'(\tilde{x}_2))^2 d\tilde{x}_2 \tag{2.13}$$

for functions $w \in C^1[a,a+\delta]$ which vanish at $\tilde{x}_2 = a$. Since $H(\Omega)$ is the completion of C_0^∞ functions, one first uses (2.13) to prove (2.12) for such functions and then takes the limit.

Since $(\tilde{v}\cdot\tilde{\nabla})\tilde{v} = -\tilde{\nabla}\tilde{p}$, it follows from (2.12) that

$$\int_{A(\varepsilon,\delta)} |\tilde{\nabla}\tilde{p}| = o(\delta) \quad \text{as} \quad \delta \to 0 \quad .$$

If $\phi \in C_0^\infty(-\varepsilon,\varepsilon)$, then

$$\int_{A(\varepsilon,\delta)} \phi'(\tilde{x}_1)\tilde{p}(\tilde{x}_1,\tilde{x}_2)d\tilde{x}_1 d\tilde{x}_2 = \int_{-\varepsilon}^{\varepsilon}\int_0^{\delta} \phi'(\tilde{x}_1)\tilde{p}(\tilde{x}_1,h(\tilde{x}_1) + \tilde{x}_2)d\tilde{x}_2 d\tilde{x}_1$$

$$= -\int_{-\varepsilon}^{\varepsilon}\int_0^{\delta} \phi(\tilde{x}_1)\left\{\frac{\partial\tilde{p}}{\partial\tilde{x}_1}(\tilde{x}_1,h(\tilde{x}_1)+\tilde{x}_2) + h'(\tilde{x}_1)\frac{\partial\tilde{p}}{\partial\tilde{x}_2}(\tilde{x}_1,h(\tilde{x}_1)+\tilde{x}_2)\right\}d\tilde{x}_2 d\tilde{x}_1$$

$$= o(\delta) \quad \text{as} \quad \delta \to 0 \quad .$$

If we divide both sides of this equation by δ and let $\delta \to 0$, there results

$$\int_{-\varepsilon}^{\varepsilon} \phi'(\tilde{x}_1)\tilde{p}(\tilde{x}_1,h(\tilde{x}_1))d\tilde{x}_1 = 0 \quad \text{for all} \quad \phi \in C_0^\infty(-\varepsilon,\varepsilon) \quad .$$

It follows that $\tilde{p}$ is a constant on Γ almost everywhere. q.e.d.

Theorem 2.2 did not use the symmetry of Ω, but we shall need it for Theorem 2.3. We now introduce some notation for the boundary components Γ_i. Without loss of generality, we may assume that Γ_1 is the 'exterior' component; that is, $\Omega \subset \text{int}\,\Gamma_1$. Since Ω is admissible, the set $\{x_2 = 0\} \cap \Gamma_i$ consists of two points $(\alpha_i,0)$ and $(\beta_i,0)$ with, say, $\alpha_i < \beta_i$. We may label the components so that $\alpha_1 < \alpha_2 < \beta_2 \cdots < \alpha_m < \beta_m < \beta_1$. Note that the sets $\{(x_1,0) : \alpha_1 < x_1 < \alpha_2\}$, $\{(x_1,0) : \beta_i < x_1 < \alpha_{i+1}\}$, $i = 2,\ldots,m-1$, and $\{(x_1,0): \beta_m < x_1 < \beta_1\}$ are all contained in Ω.

THEOREM 2.3. <u>The constants</u> C_i <u>in Theorem</u> 2.2 <u>are all equal</u>: $C_i = C$, $i = 1,\ldots,m$.

<u>Proof</u>. We shall prove that $C_1 = C_2$ since the other cases are similar. Near to the point $(\alpha_1,0)$, the component Γ_1 has the form $\{(h_1(x_2),x_2) : x_2 \in (-\delta,\delta)\}$ for some small $\delta > 0$. Here $h_1 \in C^\infty$ and $h_1(0) = \alpha_1$. Similarly, Γ_2 has the form $\{(h_2(x_2),x_2) : x_2 \in (-\delta,\delta)\}$ near to $(\alpha_2,0)$. Set $A = A(\delta) = \{(x_1,x_2) : x_1 \in (h_1(x_2),h_2(x_2)),\ x_2 \in (0,\delta)\}$, so that $A(\delta) \subset \Omega$ for all small δ. Define the total-head pressure $\tilde{\Phi} = \tilde{p} + \frac{1}{2}|\tilde{v}|^2 = \tilde{p} + \frac{1}{2}(\tilde{v}_1^2 + \tilde{v}_2^2)$, where $\tilde{v} = (\tilde{v}_1,\tilde{v}_2)$. Since $(\tilde{v}\cdot\nabla)\tilde{v} = -\nabla\tilde{p}$ (almost everywhere) in Ω, we have

$$\begin{aligned} \frac{\partial}{\partial x_1}\tilde{\Phi} &= \frac{\partial \tilde{p}}{\Phi x_1} + \tilde{v}_1\frac{\partial \tilde{v}_1}{\partial x_1} + \tilde{v}_2\frac{\partial \tilde{v}_2}{\partial x_1} \\ &= \left(-\tilde{v}_1\frac{\partial \tilde{v}_1}{\partial x_1} - \tilde{v}_2\frac{\partial \tilde{v}_1}{\partial x_2}\right) + \tilde{v}_1\frac{\partial \tilde{v}_1}{\partial x_1} + \tilde{v}_2\frac{\partial \tilde{v}_2}{\partial x_1} \\ &= -\tilde{v}_2\tilde{\omega} \quad \text{almost everywhere in } \Omega\ , \end{aligned} \tag{2.14}$$

where $\tilde{\omega} = \dfrac{\partial \tilde{v}_1}{\partial x_2} - \dfrac{\partial \tilde{v}_2}{\partial x_1}$ denotes the vorticity. Integrating this equation over $A(\delta)$ yields

$$\int_0^{\delta}\{\tilde{p}(h_2(x_2),x_2) - \tilde{p}(h_1(x_2),x_2)\}dx_2 = \delta(C_2 - C_1)$$
$$= -\int_{A(\delta)} \tilde{v}_2\tilde{\omega} \ ,$$

or, equivalently

$$|C_2 - C_1| = \frac{1}{\delta}\ |\int_{A(\delta)}\tilde{v}_2\tilde{\omega}| \ . \tag{2.15}$$

If we extend $\tilde{v}_2$ to $\mathbb{R}\times(0,\delta)$ as zero outside $A(\delta)$, then

$$\int_{A_\delta} \frac{|\tilde{v}_2(x_1,x_2)|^2}{x_2^2} = \int_{-\infty}^{\infty}\int_0^{\delta} \frac{|\tilde{v}_2(x_1,x_2)|^2}{x_2^2}\, dx_2dx_1$$
$$< 4\int_{-\infty}^{\infty}\int_0^{\delta} |\frac{\partial}{\partial x_2}\tilde{v}_2|^2 < 4\int_{A(\delta)} |\nabla\tilde{v}|^2$$

where we have used (2.13) and the fact that $\tilde{v}_2(x_1,0) = 0$ by symmetry. The use of these estimates in (2.15) give

$$|C_1 - C_2|^2 < \frac{1}{\delta^2}\left(\int_{A(\delta)} x_2^2\frac{|\tilde{v}_2|^2}{x_2^2}\right)\left(\int_{A(\delta)}\tilde{\omega}^2\right)$$
$$< \frac{1}{\delta^2}\,(4\delta^2\int_{A(\delta)}|\nabla\tilde{v}|^2)\ (4\int_{A(\delta)}|\nabla\tilde{v}|^2) = 16(\int_{A(\delta)}|\nabla\tilde{v}|^2)^2$$
$$\to 0 \quad \text{as} \quad \delta \to 0 \ .$$

Hence, $C_1 = C_2$. q.e.d.

As noted before, the use of Theorem 2.3 and (1.4) in (2.10) gives a contradication and so (1.16) holds. The Leray-Schauder theorem then gives the existence of a solution $v \in H_s(\Omega)$ to (1.13), and the existence of a pressure p and the regularity of (u,p) following in a standard way.

3. REMARKS ON THE GENERAL CASE

We now consider the problem of solving (1.1) - (1.3) in a general bounded domain $\Omega \subset \mathbb{R}^2$. We assume $\partial\Omega$ is smooth and that it consists of $m > 1$ components. Furthermore, f and $\tilde{g}$ are smooth functions from $\bar{\Omega}$ and $\partial\Omega$, respectively, to R^2. In order to prove the existence of a solution, it suffices to prove (1.16). If we assume this is false, then there is a sequence with $v_n \in S(\nu_n)$ and $|v_n|_H \to \infty$ as $n \to \infty$. Setting $\tilde{v}_n = v_n/|v_n|_H$ and $\tilde{p}_n = p_n/|v_n|_H^2$ yields $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H(\Omega)$, $\tilde{v}_n \to v$ in $L^q(\Omega)$, $q \in [1,\infty)$, $\tilde{p}_n \rightharpoonup \tilde{p}$ in $W^{1,q}(\Omega)$, $q \in (1,2)$, and $\tilde{p}_n \to \tilde{p}$ in $L^q(\Omega)$, $q \in [1,\infty)$. The pair $(\tilde{v},\tilde{p})$ is a weak solution of the steady Euler equations $(\tilde{v}\cdot\nabla)\tilde{v} = -\nabla\tilde{p}$ in Ω. Theorem 2.2 shows that $\tilde{p}$ is a constant C_i on each component Γ_i of $\partial\Omega$. In order to derive a contradiction from (2.7), it suffices to show that the C_i are all equal.

Theorem 2.3 showed that if $v \in H_s(\Omega)$ is a weak solution of the steady Euler equations in an admissible domain Ω, then the corresponding pressure $\tilde{p}$ has the same constant value on each of its components. If we could show for general Ω that

$$\tilde{v} \in H(\Omega) \text{ and } (\tilde{v}\cdot\nabla)\tilde{v} = -\nabla\tilde{p} \text{ in } \Omega \quad \text{implies} \quad \tilde{p} = C \quad \text{on} \quad \Gamma_i,\ i = 1,\dots,m\ , \tag{3.1}$$

then we would have solved (1.1) - (1.3). Unfortunately, (3.1) is not true as the following simple example shows.

EXAMPLE 3.1. <u>Let</u> $\Omega = \{(r,\theta) : 1 < r = \sqrt{x_1^2 + x_2^2} < R,\ 0 < \theta < 2\pi\}$ <u>and let</u> $\psi \in C^1[1,R]$ <u>with</u> $\psi'(1) = \psi'(R) = 0$ <u>and</u> $\psi'' \in L^2(1,R)$. <u>Define</u> $\tilde{v}(r,\theta) = (\frac{\partial}{\partial x_2}\psi(r), -\frac{\partial}{\partial x_1}\psi(r)) = (\frac{x_2}{r}\psi'(r), -\frac{x_1}{r}\psi'(r)) \in H(\Omega)$ <u>and define</u>

$$\tilde{p}(r) = \int_1^r (\psi'(w))^2/w\, dw\ ,\ r \in [1,R]\quad .$$

<u>Then</u> $(\tilde{v},\tilde{p})$ <u>is a weak solution of the steady Euler equations in</u> Ω:

$$\left.\begin{aligned}(\tilde{v}\cdot\nabla)\tilde{v} &= -\nabla\tilde{p}\ ,\\ \nabla\cdot\tilde{v} &= 0\end{aligned}\right\}\ \underline{\text{in}}\ \Omega\ ,$$

$$\tilde{v} = 0 \ \underline{\text{on}}\ \partial\Omega\quad .$$

If ψ' is not identically zero, then the pressure at $r = R$ will be strictly positive, and so (3.1) does not hold. This example suggests that the approach of section 2 is only practical when one considers symmetrical domains

Ω and symmetrical solutions $\tilde{v} \in H_s(\Omega)$. Condition (3.1) is actually too strong a requirement since $\tilde{v}$ is not just any solution of the Euler equations, but is a certain limit of solutions to the steady Navier-Stokes equations. For definiteness, we shall assume throughout the rest of this paper that Ω is the annulus of Example 3.1 and $f \equiv 0$. The boundary data $\tilde{g}$ is assumed to satisfy (1.4) but we drop the demand of symmetry about $\{x_2 = 0\}$. Recall that we begin by fixing some $\nu_o > 0$, let $S(\nu)$ be as in (1.14), and then try to prove (1.16) by assuming the contrary and deriving a contradiction. Let $\nu_n \in [\nu_o, \tilde{\nu}]$ be such that $|v_n|_H \to \infty$, where $v_n = u_n - g$. Equations (1.1) - (1.3) lead to certain maximum principles; more precisely, if $\Phi_n = p_n + \frac{1}{2}|u_n|^2$ denotes the total head pressure, then

$$\nu_n \Delta\Phi_n - u_n \cdot \nabla\Phi_n = \nu_n \omega_n^2 \quad \text{in } \Omega \quad , \tag{3.2}$$

where $\omega_n = \frac{\partial}{\partial x_2}(u_n)_1 - \frac{\partial}{\partial x_1}(u_n)_2$ denotes vorticity. Equation (3.2) yields a one-sided maximum principle for Φ_n: if A is an open subset of Ω, then Φ_n takes its maximum on $\bar{A}$ at ∂A. This maximum principle was used by Gilbarg and Weinberger [5], [6] to study the difficult problem of steady Navier-Stokes flow past a body in the plane. Define $\tilde{\Phi}_n = \Phi_n/|v_n|_H^2 = \tilde{p}_n + \frac{1}{2}|\tilde{v}_n + g/|v_n|_H|^2$, and note that $\tilde{\Phi}_n$ satisfies a one-sided maximum principle. Since $(\tilde{v}_n, \tilde{p}_n)$ converge in a certain sense to a solution $(\tilde{v}, \tilde{p})$ of the steady Euler equations, one might expect that the corresponding total head pressure $\tilde{\Phi} = \tilde{p} + \frac{1}{2}|\tilde{v}|^2$ satisfies some (weak) version of a one-sided maximum principle. We prove this is true in the following theorem and the remarks thereafter.

THEOREM 3.2. Let s,t denote arbitrary numbers satisfying $1 < s < t < R$, and set $A = \{(r,\theta) : s < r < t,\ 0 \le \theta \le 2\pi\}$. Then there exist sequences with $s_j \uparrow s$ and $t_j \downarrow t$ as $j \to \infty$ such that $\tilde{\Phi}(s_j, \cdot)$, $\tilde{\Phi}(t_j, \cdot) \in C[0,2\pi]$ and

$$\operatorname*{ess\,sup}_{(x,y)\in A} \tilde{\Phi}(x,y) \le \max_{\substack{r=s_j,t_j \\ \theta\in[0,2\pi]}} \tilde{\Phi}(r,\theta) \quad , \quad j = 1,2,\ldots \quad .$$

<u>Proof</u>. For $r \in (1,R)$ define

$$z_n(r) = \int_0^{2\pi} |\tilde{\Phi}(r,\theta) - \tilde{\Phi}_n(r,\theta)| \, |\frac{\partial}{\partial\theta} \tilde{\Phi}(r,\theta) - \frac{\partial}{\partial\theta} \tilde{\Phi}_n(r,\theta)| d\theta$$

and note that

$$\int_1^R z_n(r)dr \le \text{const.} |\tilde{\Phi}-\tilde{\Phi}_n|_{L^p(\Omega)} |\nabla(\tilde{\Phi}-\tilde{\Phi}_n)|_{L^q(\Omega)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $q \in (1,2)$. Since $\tilde{\Phi}$, $\tilde{\Phi}_n$ are bounded in $W^{1,q}(\Omega)$, $q \in (1,2)$, we may assume that $\tilde{\Phi}_n \to \tilde{\Phi}$ in $L^q(\Omega)$ for all $q \in [1,\infty)$. In particular, $z_n \to 0$ in $L^1(1,R)$ as $n \to \infty$. Since $z_n \to 0$ in measure, a suitable subsequence converges to zero uniformly almost everywhere. Given any point $s \in (1,R)$, we can find a sequence $\{s_i\}_{i=1}^{\infty}$ such that

$$s_i \uparrow s \quad \text{as} \quad i \to \infty \quad \text{and} \quad z_n(s_i) \to 0 \quad \text{as} \quad n \to \infty \quad . \tag{3.3}$$

Define

$$H_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\Phi}_n(r,\theta)d\theta$$

and similarly for $H(r)$. Since $\tilde{\Phi}_n \rightharpoonup \tilde{\Phi}$ in $W^{1,q}(\Omega)$, $q \in (1,2)$, standard results for traces show that $H_n - H \to 0$ in $C[1,R]$ as $n \to \infty$. Let i be fixed and let $\theta_n \in [0,2\pi]$ be such that

$$\tilde{\Phi}_n(s_i,\theta_n) - \tilde{\Phi}(s_i,\theta_n) = H_n(s_i) - H(s_i) \quad .$$

Now

$$\begin{aligned}
&\max_{\theta\in[0,2\pi]} |\tilde{\Phi}_n(s_i,\theta) - \tilde{\Phi}(s_i,\theta)|^2 \\
&\quad \le |\tilde{\Phi}_n(s_i,\theta_n) - \tilde{\Phi}(s_i,\theta_n)|^2 + \int_0^{2\pi} |\frac{\partial}{\partial\theta} (\tilde{\Phi}_n(s_i,\theta) - \tilde{\Phi}(s_i,\theta))^2| d\theta \qquad (3.4) \\
&\quad = |H_n(s_i) - H(s_i)|^2 + 2z_n(s_i) \to 0 \quad \text{as} \quad n \to \infty \quad .
\end{aligned}$$

A similar result holds for a suitable sequence $t_i \downarrow t$. Since $\tilde{\Phi}_n$ satisfies a one-sided maximum principle in Ω, we have

$$\hat{\Phi}_n(x,y) \le \max_{\substack{r=s_i,t_i \\ \theta\in[0,2\pi]}} \hat{\Phi}_n(r,\theta), \ (x,y) \in A \quad .$$

If we let $n \to \infty$ and use (3.4), then the theorem is proved. q.e.d.

Remark. A similar version of Theorem 3.2 holds for arbitrary open sets A with $\bar{A} \subset \Omega$. One can find open sets $\{A_j\}_{j=1}^{\infty}$ with (i) ∂A_j of class C^{∞}, (ii) $\bar{A} \subset A_j \subset \bar{A}_j \subset \Omega$, (iii) $\hat{\Phi}$ is continuous on ∂A_j, (iv) $\bar{A}_k \subset A_j$ if $k > j$, (v) $A_j \to A$ in the sense that $A = \text{int} \cap A_j$, and such that

$$\operatorname*{ess\,sup}_{A} \hat{\Phi} \le \max_{\partial A_j} \hat{\Phi}, \ j = 1,2,\ldots \quad .$$

Example 3.1 gives a whole family of solutions to the steady Euler equations in Ω for which the pressure is not the same constant value on $\partial\Omega$. We claim that such flows cannot arise as the limit of unbounded solutions to (1.1) - (1.3) with $f \equiv 0$ and ν_o replaced by ν_n. Indeed, let ψ be as in Example 3.1, and note that the total head pressure is only a function of r:

$$\hat{\Phi}(r) = \int_1^r (\psi'(w))^2/w \, dw + \frac{1}{2}|\psi'(r)|^2, \ r \in [1,R] \quad .$$

Since $\psi'' \in L^2(1,R)$ with $\psi'(1) = \psi'(R) = 0$ (which is equivalent to $\tilde{v} = (\psi_y,-\psi_x) \in H(\Omega))$ we know that $\hat{\Phi}$ is (Hölder) continuous on $[1,R]$. Assume for the moment that $\hat{\Phi}$ satisfies Theorem 3.2 and ψ' is not identically zero on $[1,R]$. Then $\hat{\Phi}$ takes its maximum on any interval $[s,t]$, with $1 \le s < t \le R$, at one or both of the endpoints. Since $\hat{\Phi}(1) = 0$ and $\hat{\Phi} \ge 0$ on $[1,R]$, it follows that $\hat{\Phi}$ is non-decreasing on $[1,R]$. A calculation gives

$$\hat{\Phi}(t) - \hat{\Phi}(s) = \int_s^t \frac{\psi'(w)}{w} (w\psi'(w))' dw \quad ,$$

for any $s,t \in [1,R]$ whence

$$0 \le \psi'(w)(w\psi'(w))' \text{ almost everywhere on } [1,R] \quad . \tag{3.5}$$

Since ψ' is not identically zero on $[1,R]$ by hypothesis, there exists some $w_o \in (1,R)$ such that $\psi'(w_o) \ne 0$. Let B denote the largest open interval containing w_o such that $\psi' \neq 0$ on B. Since $\psi'(1) = \psi'(R) = 0$, we know that ψ' vanishes at the endpoints of B. On the other hand, (3.5) says that $(w\psi'(w))'$ is one signed (almost everywhere) on B. The only possibility is that $(w\psi'(w))' = 0$ on B, whence $\psi' = 0$ on B. This is a contradiction, and so the non-trivial solutions $(\hat{v},\hat{p})$ of Example 3.1 do not satisfy Theorem 3.2. Therefore, it is natural to ask the follow question:

> Let $\Omega \subset R^2$ denote an annulus and let $(\hat{v},\hat{p})$ be a weak solution of the Euler equations $(\hat{v}\cdot\nabla)\hat{v} = -\nabla\hat{p}$ in Ω with $\hat{v} \in H(\Omega)$. If $\hat{\Phi} = \hat{p} + \frac{1}{2}|\hat{v}|^2$ satisfies Theorem 3.2 and the remark thereafter, then does $\hat{p}|_{r=1} = \hat{p}|_{r=R}$, or, equivalently, does $\hat{\Phi}|_{r=1} = \hat{\Phi}|_{r=R}$?

There are other maximum principles associated with (1.1) - (1.3) with ν_o replaced by ν and $f \equiv 0$. If we denote the vorticity by $\omega = \frac{\partial}{\partial x_2} u_1 - \frac{\partial}{\partial x_1} u_2$, then a calculation gives

$$-\nu\Delta\omega + u\cdot\nabla\omega = 0 \quad \text{in } \Omega \quad ,$$

so that ω satisfies a two-sided maximum principle. If $\{u_n\}_{n=1}^{\infty}$ are solutions of (1.1) - (1.3) with ν_o replaced by ν_n, then

$$\hat{\omega}_n = \frac{1}{|v_n|_H}\left\{\frac{\partial}{\partial x_2}(u_n)_1 - \frac{\partial}{\partial x_1}(u_n)_2\right\}$$

satisfies a two-sided maximum principle in Ω: if A is an open subset of Ω, then the maximum and minimum values of $\hat{\omega}_n$ on $\bar{A}$ occur on ∂A. Although $\hat{\omega}_n$ is bounded in $L^2(\Omega)$ and converges weakly to $\hat{\omega} = \frac{\partial}{\partial x_2}\hat{v}_1 - \frac{\partial}{\partial x_1}\hat{v}_2$, this is not enough to prove results analogous to Theorem 3.2 for $\hat{\omega}$; we would need some control on derivatives of $\hat{\omega}_n$.

If $(\hat{v},\hat{p})$ is a weak solution to the steady Euler equations in Ω with $\hat{v} \in H(\Omega)$ (and $\hat{v}$ is not necessarily a limit of Navier-Stokes solutions), then the total head pressure $\hat{\Phi} \in W^{1,q}(\Omega)$, $q \in (1,2)$, whence $\hat{\Phi} \in L^q(\Omega)$ for all $q \in [1,\infty)$. By using the averaging methods of Gilbarg and Weinberger [5],

[6] for the pressure $\tilde{p}$, one can show that $\tilde{p} \in C(\bar{\Omega})$ and $\tilde{v}(x,y) \to 0$ as $(x,y) \to \partial\Omega$. If we know in addition that Theorem 3.2 and the remark after it hold, then we have $\tilde{\phi} \in L^{\infty}(\Omega)$. It is not known if $\tilde{\phi} \in C(\bar{\Omega})$.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Navier-Stokes equations, a priori bounds, maximum principles

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper concerns the existence of steady solutions to the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$. The condition of solenoidality for the velocity field imposes a necessary condition on the boundary data. For a certain class of symmetrical domains, we show that this necessary condition implies the existence of a solution to the problem. The method consists of proving a priori bounds on solutions by assuming the contrary, rescaling the equations, and then arriving at a solution to the steady Euler equations in the limit. Examination of this equation leads to the desired contradiction. After

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ABSTRACT (continued)

one has suitable bounds on any solutions, one uses the Leray-Schauder theorem to prove existence.

In addition, we remark on the problem of a general bounded domain $\Omega \subset \mathbb{R}^2$, and suggest how certain maximum principles might yield the expected results.

FILME

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